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Generators for nonlinear canonical transformations

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Abstract. Canonical transformations whose generators are linear in the basic operators P_j and arbitrary in the canonically conjugate operators Q_j are explicitly constructed. It is shown that they correspond to gauge transformations and changes of variables. Some applications are mentioned in the one-dimensional case.

1. Introduction

From the beginning of quantum mechanics, many works have been devoted to canonical transformations [1]. The first ones were of course the linear transformations [2]. The study of nonlinear transformations came afterwards. The analysis was essentially concentrated on some classes of nonlinear bijective [3, 4] and non-bijective [5] transformations. The algebra of the generators was only considered for linear transformations. In this paper, we want to show that the canonical transformations whose generators $f(Q_1, \ldots, Q_n)$ and $g(Q_1, \ldots, Q_n)P_i$ are written as functions of n couples of canonical conjugate operators Q_j and P_j form an infinite-dimensional group composed of gauge transformations and changes of variables. Furthermore, we will show that we are able to build operators that realize these transformations. This is very interesting in many fields but particularly in the case of exactly solvable models [6–9]. This is also very useful in the construction of realizations of Lie algebras [10–12] and in the study of special functions connected to them [13–15]. Here, we essentially want to put the accent on the algebraic aspect of the problem. In a first section, we begin by introducing and studying the algebra of the generators. The group of canonical transformations will be examined in the next section. Finally, in the last section, we will give some applications in the case of one degree of freedom.

2. Algebra of the generators

The canonical transformations that will be discussed in this paper are in fact transformations for which we give the generators. They will be identified as canonical transformations afterwards. Let us first introduce n couples of canonical conjugate operators Q_j and P_j ($j = 1, \ldots, n$) that obey the usual commutation relations

$$[Q_j, Q_k] = 0 \qquad [P_j, P_k] = 0 \qquad [Q_j, P_k] = \mathrm{i}\delta_{jk} \qquad \text{for } j, k = 1, \ldots, n.$$

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The first generators we consider are those given by any function g of the basic operators Q_j

$$g(Q_1, \ldots, Q_n). \tag{2.1}$$

We suppose that this function g is expandable in Taylor's or Laurent's series. It is clear that they define an infinite-dimensional Abelian algebra because the commutator of two such generators is identically null:

$$[g(Q_1, \ldots, Q_n), g'(Q_1, \ldots, Q_n)] = 0. \tag{2.2}$$

A basis for this algebra can be chosen to be the set of all monomials

$$Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} \tag{2.3}$$

where $m_1, \ldots, m_n = 0, 1, 2, \ldots$ or $0, \pm 1, \pm 2, \ldots$ depending on whether we take functions expandable in Taylor's or Laurent's series.

Let us now add to these generators other generators given by linear expressions in the operators P_j

$$\sum_{j=1}^{n} f_j(Q_1, \ldots, Q_n) P_j \tag{2.4}$$

where the functions f_j have the same properties as the function g previously introduced. The algebra is no longer Abelian but closes with respect to the commutator. We have indeed

$$\left[\sum_j f_j(Q_1, \ldots, Q_n) P_j, \sum_k f'_k(Q_1, \ldots, Q_n) P_k\right] = -\mathrm{i} \sum_{j,k} \left(f_j \frac{\partial f'_k}{\partial Q_j} - f'_j \frac{\partial f_k}{\partial Q_j}\right) P_k. \tag{2.5}$$

For this algebra we can also introduce a basis similar to (2.3)

$$Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} P_i \tag{2.6}$$

where $m_1, \ldots, m_n = 0, 1, 2, \ldots$ or $0, \pm 1, \pm 2, \ldots$ and $i = 1, \ldots, n$.

Finally, we may combine both types of generators and the whole algebra is still closed with respect to the commutator as can easily be seen from the commutator

$$\left[\sum_j f_j(Q_1, \ldots, Q_n) P_j, g(Q_1, \ldots, Q_n)\right] = -\mathrm{i} \sum_j f_j \frac{\partial g}{\partial Q_j}. \tag{2.7}$$

In the following, we will denote by $\mathscr{A}_0$ the algebra generated by (2.1), by $\mathscr{A}_1$ the algebra generated by the operators (2.4) and by $\mathscr{A}$ the whole algebra containinig both types of generators. It is clear from the commutators (2.2), (2.5) and (2.7), that $\mathscr{A}_0$ and $\mathscr{A}_1$ are subalgebras of $\mathscr{A}$ and moreover that $\mathscr{A}_0$ is an invariant subalgebra of $\mathscr{A}$: $[\mathscr{A}, \mathscr{A}_0] \subset \mathscr{A}_0$.

3. Canonical transformations

From the algebra we can obtain the elements of the group by using the usual procedure of exponentiating the algebra. This gives for the generators of $\mathscr{A}_0$ the following elements

$$\mathrm{e}^{\mathrm{i} g(Q_1, \ldots, Q_n)}$$

where for the sake of commodity, we introduce the factor i. Let us now look a little more at the associated transformations. For this, we will use the Baker–Hausdorff formula

$$\mathrm{e}^{A} B\, \mathrm{e}^{-A}=\sum_{m=0}^{\infty} \frac{1}{m!} A^{m}\{B\} \tag{3.1}$$

where

$$A^{m}\{B\}=[A, A^{m-1}\{B\}]=[A,[A, \ldots[A, B] \ldots]]$$

to get

$$\begin{aligned} \bar{Q}_j &= \mathrm{e}^{\mathrm{i} g(Q_1, \ldots, Q_n)} Q_j\, \mathrm{e}^{-\mathrm{i} g(Q_1, \ldots, Q_n)}=Q_j \\ \bar{P}_j &= \mathrm{e}^{\mathrm{i} g(Q_1, \ldots, Q_n)} P_j\, \mathrm{e}^{-\mathrm{i} g(Q_1, \ldots, Q_n)}=P_j-\frac{\partial g}{\partial Q_j} . \end{aligned} \tag{3.2}$$

We recognize of course very usual transformations in physics known as gauge transformations. These are thus the transformations generated by the elements of the algebra $\mathscr{A}_0$. For the generators of $\mathscr{A}_1$, we proceed in a similar fashion. The basic operators Q_j are transformed according to

$$\bar{Q}_j=\exp \left(\mathrm{i} \sum_k f_k(Q_1, \ldots, Q_n) P_k\right) Q_j \exp \left(-\mathrm{i} \sum_k f_k(Q_1, \ldots, Q_n) P_k\right)=\sum_{m=0}^{\infty} \frac{1}{m!} A^{m}\{Q_j\} \tag{3.3}$$

where A is here a linear operator given by

$$A=\mathrm{i} \sum_k f_k(Q_1, \ldots, Q_n) P_k .$$

Let us denote by $h(Q_1, \ldots, Q_n)$ any function of $Q_1, \ldots, Q_n$. It is then very easy to show that

$$[A, h(Q_1, \ldots, Q_n)]=\sum_k f_k(Q_1, \ldots, Q_n) \frac{\partial h(Q_1, \ldots, Q_n)}{\partial Q_k} .$$

Applying this last equation m times, we get

$$A^{m}\{h(Q_1, \ldots, Q_n)\}=\left(\sum_k f_k(Q_1, \ldots, Q_n) \frac{\partial}{\partial Q_k}\right)^{m} h(Q_1, \ldots, Q_n). \tag{3.4}$$

In particular, if we apply formula (3.4) to equation (3.3), we get the final expression

$$\begin{aligned} \bar{Q}_j &= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum_k f_k(Q_1, \ldots, Q_n) \frac{\partial}{\partial Q_k}\right)^{m} Q_j \\ &= \exp \left(\sum_k f_k(Q_1, \ldots, Q_n) \frac{\partial}{\partial Q_k}\right) Q_j . \end{aligned} \tag{3.5}$$

The new operators $\bar{Q}_j$ are given as a function of the old ones, Q_j, by acting with a hyperdifferential operator. The result is a given function depending on the $Q_1, \ldots, Q_n$ only. The canonical transformations generated by (2.4) are thus changes of variables. We will write them as

$$\bar{Q}_j=u_i(Q_1, \ldots, Q_n). \tag{3.6}$$

We still have to determine the law of transformation of the basic operators P_j. The problem is a little more complex and we will proceed in a different indirect way. First we look for the general form of $\bar{P}_j$. Afterwards, we fix completely the $\bar{P}_j$ by taking into account of the fact that they are canonically conjugate to the $\bar{Q}_j$ as the operators P_j are with respect to the Q_j. The first step in the application of Baker-Hausdorff formula (3.1) is

$$[A, P_j] = -\sum_k \frac{\partial f_k(Q_1, \ldots, Q_n)}{\partial Q_j} P_k$$

that will be rewritten as

$$[A, P_j] = \sum_k g_{jk}^{(1)}(Q_1, \ldots, Q_n) P_k.$$

A repeated application of this formula will give something like

$$A^m\{P_j\} = \sum_k g_{jk}^{(m)}(Q_1, \ldots, Q_n) P_k \tag{3.7}$$

where the functions $g_{jk}^{(m)}(Q_1, \ldots, Q_n)$ are not easy to obtain but are well-defined functions. The law of transformation of the P_j can be summarized by the following equations:

$$\begin{aligned}\bar{P}_j &= \exp\left(\mathrm{i}\sum_k f_k(Q_1, \ldots, Q_n) P_k\right) P_j \exp\left(-\mathrm{i}\sum_k f_k(Q_1, \ldots, Q_n) P_k\right) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} A^m\{P_j\} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_k g_{jk}^{(m)}(Q_1, \ldots, Q_n) P_k \\ &= \sum_k g_{jk}(Q_1, \ldots, Q_n) P_k.\end{aligned} \tag{3.8}$$

Consequently, we point out that $\bar{P}_j$ is a linear conbination of the operators P_j with coefficients given by functions of $Q_1, \ldots, Q_n$. Let us now impose on this expression the commutation relations

$$[\bar{Q}_j, \bar{P}_k] = \mathrm{i}\delta_{jk} \qquad \text{and} \qquad [\bar{P}_j, \bar{P}_k] = 0. \tag{3.9}$$

The first commutator, when replacing $\bar{Q}_j$ and $\bar{P}_j$ by their value (3.6) and (3.8) respectively, becomes

$$\left[u_j(Q_1, \ldots, Q_n), \sum_l g_{kl}(Q_1, \ldots, Q_n) P_l\right] = \mathrm{i}\delta_{jk}$$

or

$$\sum_l g_{kl} \frac{\partial u_j}{\partial Q_l} = \delta_{jk} \tag{3.10}$$

by expanding the commutator. It is very useful to introduce for the following the matrix $\mathbf{v}$ whose elements are given by

$$v_{jk} = \frac{\partial u_k}{\partial Q_j}. \tag{3.11}$$

Relation (3.10) points out the fact that the matrix $\mathbf{g}$, whose elements are the functions g_{kl} introduced in (3.8), is the inverse matrix of $\mathbf{v}$:

$$g_{kl} = v_{kl}^{-1}. \tag{3.12}$$

This of course can be realized only if $\mathbf{v}^{-1}$ exists, that is to say if

$$\det|v_{jk}| = \det\left|\frac{\partial u_k}{\partial Q_j}\right| \neq 0.$$

This condition is precisely what is necessary to ensure the independence of the new operators $\bar{Q}_j$. Finally, it is easy to verify that the other commutation relation in (3.9) is automatically verified, provided we suppose

$$\frac{\partial^2 u_j}{\partial Q_k\,\partial Q_l} = \frac{\partial^2 u_j}{\partial Q_l\,\partial Q_k}.$$

The final form of this second type of canonical transformations is given by

$$\begin{aligned} \bar{Q}_j &= u_i(Q_1, \ldots, Q_n) \\ \bar{P}_j &= \sum_k v_{jk}^{-1}(Q_1, \ldots, Q_n)P_k \end{aligned} \tag{3.13}$$

where v_{jk} is given by (3.11). If we now want to consider the complete algebra $\mathscr{A}$, we have to combine both types of transformations. We will use the property that all transformation associated with $\mathscr{A}$ is the product of a transformation related with $\mathscr{A}_1$ and a transformation of $\mathscr{A}_0$. This gives the general form

$$\begin{aligned} \bar{Q}_j &= u_i(Q_1, \ldots, Q_n) \\ \bar{P}_j &= \sum_k v_{jk}^{-1}(Q_1, \ldots, Q_n)\left[P_k - \frac{\partial g(Q_1, \ldots, Q_n)}{\partial Q_k}\right] \end{aligned} \tag{3.14}$$

because it is always possible to factorize the transformation operator in the following way:

$$\begin{aligned} &\exp\left(\mathrm{i}\sum_k f_k(Q_1, \ldots, Q_n)P_k + g(Q_1, \ldots, Q_n)\right) \\ &\qquad = \exp\left(\mathrm{i}\sum_k f'_k(Q_1, \ldots, Q_n)P_k\right)\exp(\mathrm{i}g'(Q_1, \ldots, Q_n)) \end{aligned} \tag{3.15}$$

where $f'_k(Q_1, \ldots, Q_n)$ and $g'_k(Q_1, \ldots, Q_n)$ are well-defined functions obtained from $f_k(Q_1, \ldots, Q_n)$ and $g_k(Q_1, \ldots, Q_n)$. In the following, we will always use this factorized form (3.15). We want now to mention that in this paper we start from the algebra of the generators and then go to the group and to the identification of the transformations. It is also possible to proceed in the opposite direction. Let us show this for the changes of variables. We will use a procedure analogous to that used by Ogievetsky [16] for the general covariance group. Let us start from equation (3.13) that can also be written as

$$\bar{Q}_j = Q_j + \sum_{m_i} \varepsilon^{(j)}_{m_1 m_2 \ldots m_n} Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} \tag{3.16}$$

by expanding the functions $u_j(Q_1, \ldots, Q_n)$ in Taylor's series. We choose the $\varepsilon^{(j)}_{m_1 m_2 \ldots m_n}$ in such a manner that they are equal to zero when the transformation is the identity. This form suggests we take the $\varepsilon^{(j)}_{m_1 m_2 \ldots m_n}$ as an infinity of parameters defining the transformation. Proceeding in the same fashion as for the Lie finite-dimensional groups, expression (3.16) may be written

$$\bar{Q}_j = Q_j + \sum_{k,m_i} \varepsilon^{(k)}_{m_1 m_2 \ldots m_n} \delta_{kj} Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} = \mathrm{e}^{\mathrm{i}} X Q_j \, \mathrm{e}^{-\mathrm{i}X}$$

where $X = \Sigma_{k,m_i} \varepsilon^{(k)}_{m_1 m_2 \ldots m_n} X^{(k)}_{m_1 m_2 \ldots m_n}$ and the operators $X^{(k)}_{m_1 m_2 \ldots m_n}$ are the generators. In the case of infinitesimal transformations, these relations give

$$\mathrm{i}[X, Q_j] = \sum_{k,m_i} \varepsilon^{(k)}_{m_1 m_2 \ldots m_n} \delta_{kj} Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n}$$

or

$$\mathrm{i}[X^{(k)}_{m_1 m_2 \ldots m_n}, Q_j] = \delta_{kj} Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n}.$$

It is indeed very easy to verify that these equations admit

$$X^{(j)}_{m_1 m_2 \ldots m_n} = Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} P_j \tag{3.17}$$

as a solution. We meet again the generators we proposed at the beginning of this section. The analysis we made for changes of variables suggests that for general canonical transformations

$$\bar{Q}_j = u_j(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$$

the generators should be given by

$$Q_1^{m_1} Q_2^{m_2} \ldots Q_n^{m_n} P_1^{m'_1} P_2^{m'_2} \ldots P_n^{m'_n}.$$

4. Applications

In this section, we will examine the special situation where the dimension is equal to one. The gauge transformations (3.2) become in this case

$$\begin{aligned} \bar{Q} &= \mathrm{e}^{\mathrm{i}g(Q)} Q \, \mathrm{e}^{-\mathrm{i}g(Q)} = Q \\ \bar{P} &= \mathrm{e}^{\mathrm{i}g(Q)} P \, \mathrm{e}^{-\mathrm{i}g(Q)} = P - \frac{\mathrm{d}q}{\mathrm{d}Q}. \end{aligned} \tag{4.1}$$

For the changes of variables we have also simplifications, and [3.13] becomes

$$\begin{aligned} \bar{Q} &= \mathrm{e}^{\mathrm{i}f(Q)P} Q \, \mathrm{e}^{-\mathrm{i}f(Q)P} = \mathrm{e}^{A} Q = u(Q) \\ \bar{P} &= \mathrm{e}^{\mathrm{i}f(Q)P} P \, \mathrm{e}^{-\mathrm{i}f(Q)P} = \left[\frac{\mathrm{d}u(Q)}{\mathrm{d}Q}\right]^{-1} P \end{aligned} \tag{4.2}$$

where A is now, following (3.5), the differential operator $f(Q)\, \mathrm{d}/\mathrm{d}Q$. The link between $f(Q)$ and $u(Q)$, in this one-dimensional case, can be made explicit. Let us define

$$u(Q, t) = \mathrm{e}^{tA} Q \tag{4.3}$$

which reduces to $u(Q)$ when $t = 1$. It is easy to verify that the function $u(Q, t)$ is a solution of the partial differential equation

$$\frac{\partial u(Q, t)}{\partial t} = f(Q) \frac{\partial u(Q, t)}{\partial Q} \tag{4.4}$$

that we obtain by differentiating (4.3) with respect to t. The solution that we look for is constrained to the initial value $u(Q,0)=Q$. The system of ordinary differential equations associated with (4.4) admits the two following characteristic curves

$$u=c_1 \qquad \text{and} \qquad t+F(Q)=c_2 \tag{4.5}$$

where $F(Q)$ is a primitive of $1/f(Q)$, i.e. $\mathrm{d}F(Q)/\mathrm{d}Q=1/f(Q)$. The general solution of (4.4) is thus

$$u(Q,t)=\mathscr{F}(t+F(Q))$$

where $\mathscr{F}$ denotes an arbitrary function. If we now take into account the initial condition, we obtain

$$u(Q,t)=F^{-1}(t+F(Q))$$

which finally gives

$$u(Q)=F^{-1}(1+F(Q)) \tag{4.6}$$

where F^{-1} denotes the inverse function of $F(Q)$ introduced in (4.5). This supposes of course that this function is bijective. We thus establish the connection between the functions $u(Q)$ and $f(Q)$. The above relation (4.6) is of course more practical going from $f(Q)$ to $u(Q)$. Let us look now at some interesting special cases. When $f(Q)=KQ^n$, we obtain

$$u(Q)=\begin{cases} Q[1+K(1-n)Q^{n-1}]^{1/(1-n)} & n\neq 1 \\ \mathrm{e}^K Q & n=1. \end{cases} \tag{4.7}$$

This last case corresponds to the subgroup of dilatations. Two other special cases are interesting. They correspond respectively to $n=0$ for the translations $u(Q)=Q+K$ and to $n=2$ for the homographic transformations $u(Q)=Q/(1-KQ)$. It is remarkable that these three special cases $n=0,1,2$, whose generators are respectively P, QP and Q^2P, form a subalgebra $\mathrm{SL}(2,\mathbb{R})$ of $\mathscr{A}_1$.

To stay in the realizations of $\mathrm{SL}(2,\mathbb{R})$, let us mention another application of these canonical transformations developed here. From all the realizations of $\mathrm{SL}(2,\mathbb{R})$ as a subalgebra of $\mathscr{A}$, let us show that there exists one for which one of the generators, say X, can be brought, with the help of the canonical transformations (4.1) and (4.2), to the operator P. If the realization indeed belongs to $\mathscr{A}$, we necessarily have

$$X=h_1(Q)P+h_2(Q).$$

Now, from (4.1) and (4.2), we have the general transform of P given by

$$\mathrm{e}^{\mathrm{i}f(Q)P}\,\mathrm{e}^{\mathrm{i}g(Q)}P\,\mathrm{e}^{-\mathrm{i}g(Q)}\,\mathrm{e}^{-\mathrm{i}f(Q)P}=\left[\frac{\mathrm{d}u(Q)}{\mathrm{d}Q}\right]^{-1}P-\left[\frac{\mathrm{d}u(Q)}{\mathrm{d}Q}\right]^{-1}\frac{\mathrm{d}g}{\mathrm{d}Q}$$

from which it is obvious to identify

$$\frac{\mathrm{d}u(Q)}{\mathrm{d}Q}=1/h_1(Q) \qquad \text{and} \qquad \frac{\mathrm{d}g(Q)}{\mathrm{d}Q}=h_2(Q)/h_1(Q).$$

After solving these differential equations, we finally obtain $u(Q)$ and $g(Q)$ and thus completely characterize the canonical transformation. This is particularly interesting in the construction of exactly solvable models [6–9]. Another field of application where these transformations play a big role is the field of special functions [10–12]. In many cases where special functions were involved, they generally appear in the matrix elements of a representation so that the realization in itself has no special meaning. In those cases, of course, it is judicious to use canonical transformations in order to simplify at most the realization itself.

The last application we want to mention here concerns the canonical transformations themselves. In some cases, it is easier to work with the generators and their commutation relations than with the elements of the group. We can see for example that, if we combine gauge transformations and linear canonical transformations [2], we generate the whole group of canonical transformations [4]. As we saw previously, indeed, the generators of gauge transformations are given by any function $g(Q)$ while linear transformations are generated by P^2, QP and Q^2. To close the algebra, we are indeed obliged to include changes of variables. The commutation $[P^2, u(Q)]$ introduces in fact other generators of the form $f(Q)P$. Proceeding step by step, we finally see that we will have to include any generator of the form $f(Q)P^n$ and this, of course, generates the whole group.

5. Conclusions

We have explicitly constructed the canonical transformations whose generators have a linear dependency in the bassic operators P_j and an arbitrary dependency in the Q_j. We obtained in this way gauge transformations and changes of variables. We thus analysed a subgroup of the general canonical transformations group. If we exclude the linear transformations, the subgroup we have considered here is probably the simplest subgroup of the general group. It seems indeed very difficult, except in very special cases, to combine other generators without having to include the set of all the generators in order to close the algebra with respect to the commutator. A field where it is very interesting to apply such transformations, as those studied here, is the construction of realizations of a semisimple Lie algebra.

References

[1] Dirac P A M 1947 *The Principles of Quantum Mechanics* (Oxford: Clarendon)
[2] Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
[3] Moshinsky M, Seligman T H and Wolf K B 1972 *J. Math. Phys.* **13** 901
Mello P A and Moshinsky M 1975 *J. Math. Phys.* **16** 2017
[4] Colegrave R K and Ponnudarai Bala 1985 *J. Phys. A: Math. Gen.* **18** 779
[5] Kramer P, Moshinsky M and Seligman T H 1978 *J. Math. Phys.* **19** 683
Deenen J 1981 *J. Phys. A: Math. Gen.* **14** L273
[6] Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2229
[7] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
[8] Deenen J 1990 *J. Phys. A: Math. Gen.* **23** 133
[9] Brajamani S and Singh C A 1990 *J. Phys. A: Math. Gen.* **23** 3421
[10] Joseph A 1972 *J. Math. Phys.* **13** 351
[11] Simoni A and Zaccaria F 1968 *Nuovo Cimento* A **59** 280
[12] Hong-Chen Fu and Chang-Pu Sun 1990 *J. Math. Phys.* **31** 287
Hong-Chen Fu and Chang-Pu Sun 1990 *J. Math. Phys.* **31** 2797
[13] Miller W 1968 *Lie Theory and Special Functions* (New York: Academic)
[14] Vilenking N Ja 1969 *Fonctions Spéciales et Théorie de la Représentation des Groupes* (Paris:Dunod)
[15] Wu J and Alhassid Y 1990 *J. Math. Phys.* **31** 557
[16] Ogievetsky V I 1973 *Lett. Nuovo Cimento* **8** 988